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Interpolation Systems in R^k

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In a previous paper (*Numer. Math.* **39** (1982), 1–14), M. Gasca and J. I. Maeztu used a geometrical method for the construction of the solutions of certain Hermite and Lagrange interpolation problems in R^k. In the present paper, the method is generalized in two different ways: first, the interpolant is not assumed to be a polynomial, and second, a parameter is introduced in order to render the method more versatile.

1. INTRODUCTION

It is well known that Newton's representation of the Lagrange or Hermite interpolation polynomial in one variable stems from the fact that the polynomials

$$\left.\begin{aligned} &\psi_i(x)=\prod_{j=0}^{i-1}(x-x_j), \qquad i=1,2,\dots,n, \\ &\psi_0(x)=1 \end{aligned}\right\} \tag{1.1}$$

satisfy

$$\psi_i(x_j)=0 \qquad \text{if} \quad i>j. \tag{1.2}$$

This combined with a geometrical approach, enabled Gasca and Maeztu [3] to solve a large number of Lagrange and Hermite interpolation problems in several variables. The present paper relies heavily on the material in [3] and,

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therefore, we recall briefly the main ideas of that article. This is best achieved by means of an example. Consider five points in R^2, such that three of them, $u_{0j}, j=0, 1, 2$, lie on a straight line r_0, with equation $r_0=0$, and the remaining two, $u_{1j}, j=0, 1$, lie on a different straight line r_1, with equation $r_1=0$. For each point u_{ij}, let $r_{ij}=0$ be the equation of a straight line whose intersection with r_i is u_{ij} (Fig. 1). If the indices (i,j) are ordered lexicographically, the functions $\psi_{00}=1$, $\psi_{01}=r_{00}$, $\psi_{02}=r_{00}r_{01}$, $\psi_{10}=r_0$, $\psi_{11}=r_0 r_{10}$ satisfy the following property

$$\psi_{ij}(u_{hk})=0 \qquad \text{if} \quad (i,j)>(h,k), \tag{1.3}$$

which is analogous to (1.2). Furthermore, if

$$\forall (i,j) \leftrightarrow \psi_{ij}(u_{ij}) \neq 0 \tag{1.4}$$

(this is the case depicted in Fig. 1), then the Lagrange interpolation problem associated with the data $f(u_{00}), f(u_{01}), f(u_{02}), f(u_{10}), f(u_{11})$ and the space

$$\operatorname{span}\{\psi_{00}, \psi_{01}, \psi_{02}, \psi_{10}, \psi_{11}\}$$

possesses a unique solution obtainable by a Newton-like recurrence.

In a more general setting, one deals with straight lines r_i, r_{ij} and the functions ψ_{ij} are given by

$$\psi_{ij}=\prod_{s=0}^{i-1} r_s \prod_{t=0}^{j-1} r_{it} \tag{1.5}$$

(as usual, a product with an empty set of indices is taken to be one).

The situation where some of the points u_{ij} on the line r_i or some of the lines r_i coincide, may be dealt with as a limit case of the simple situation above. In analogy with the one-dimensional case that limit originates Hermite problems.

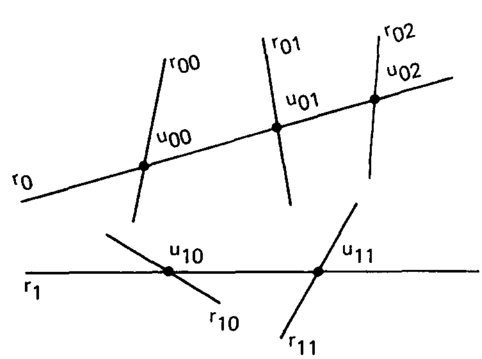

FIGURE 1

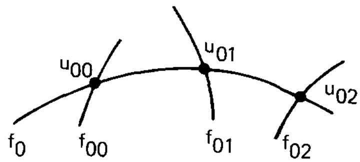

FIGURE 2

In this paper the straight lines $r_i = 0$, $r_{ij} = 0$ are replaced by curves $f_i = 0$, $f_{ij} = 0$, where f_i, f_{ij} are not necessarily polynomial. The "curved" case is more difficult to analyse than its "straight" counterpart. On the one hand the number of possible situations is considerably enlarged. On the other hand the fact that the second derivatives of f_i, f_{ij} are not identically zero causes additional problems. In order to guarantee (1.3), the functions ψ_{ij} are defined by the formula

$$\psi_{ij} = \prod_{s=0}^{i-1} f_s^{e_s} \prod_{t=0}^{j-1} f_{it}, \tag{1.6}$$

where the exponents e_s are positive integers. Figures 2 and 3 show how the coincidence of two points u_{ij} on the curve f_i gives rise to the inclusion of the derivative in the direction of the tangent to f_i as an interpolation datum.

There is another way in which the present article enlarges the versatility of the method in [3]. This extension concerns the choice of the interpolation space V. For instance, with the procedure of [3] the constant functions always belong to V, since ψ_{00} is always the function 1. The question emerges of whether it is possible to allow more flexibility in the choice of V. We answer this question in the affirmative. To illustrate this point, assume that in the situation of the figure 1 we wish to interpolate at u_{01}, u_{02}, u_{10}, u_{11} (i.e., the datum at u_{00} is removed) and we wish that V does not necessarily contain the constant functions (i.e., ψ_{00} is removed from the basis of V). In the context of the present paper it is possible to achieve those goals by setting a parameter α_{ij} with $\alpha_{00} = 0$ and $\alpha_{01} = \alpha_{02} = \alpha_{10} = \alpha_{11} = 1$. Note that the point u_{00} still plays a role in the construction of ψ_{ij} for $(i, j) \neq (0, 0)$, because of the factors f_0, f_{00}.

An easy example of the wider range of applicability of the present

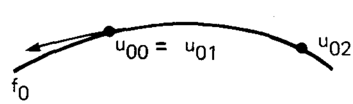

FIGURE 3

technique when compared with that of [3] is given by the problem whose only datum is

$$\frac{\partial f}{\partial x}(0, 0).$$

This problem, solvable here, cannot be dealt with within the framework of [3], where $V = \{\text{constants}\}$ regardless of the choice of r_i, r_{ij}.

Finally we study the relevance of the idea in [3] in the derivation of error bounds.

An outline of the paper is as follows. Section 2 and 3 are devoted to the presentation of the notations employed and to the definition of interpolation system in R^2. This is a set of quadruples $(f_i, f_{ij}, u_{ij}, \alpha_{ij})$, from which the basis $\{\psi_{ij}\}$ of an interpolation space V and a set of interpolation data $\{L_{hk}\}$ are derived in such a way that the associated problem is unisolvent and can be solved by a Newton-like recurrence. This recurrence is presented in Section 4, together with a remainder theory. Section 5 is devoted to some refinements and Section 6 to the extension to R^k, $k \neq 2$, which presents no further difficulty.

Examples are given throughout the paper. Additional examples together with a more detailed treatment of the present material can be seen in [6] but the omited proofs are rather simple. The reader is referred to [2] for other applications of the method. For related material see [1, 4, 5, 7, 8].

Most but not all (cf. [3]) Lagrange or Hermite two dimensional problems arise from an interpolation system. Note that the present technique starts with an interpolation system and then generates the interpolation problem.

2. Definitions and Notations

Let $\rho = (a, b) \neq (0, 0)$ be a vector in R^2. If f is sufficiently regular at the point u we denote

$$\left.\begin{aligned} \left.\frac{\partial f}{\partial \rho}\right|_u &= a \left.\frac{\partial f}{\partial x}\right|_u + b \left.\frac{\partial f}{\partial y}\right|_u \\ \left.\frac{\partial^k f}{\partial \rho^k}\right|_u &= \frac{\partial}{\partial \rho}\left(\frac{\partial^{k-1} f}{\partial \rho^{k-1}}\right)\Bigg|_u, \qquad k = 2, 3, \ldots . \end{aligned}\right\} \tag{2.1}$$

Definition 1. An interpolation system in R^2 is a set

$$S = \{(f_i, f_{ij}, u_{ij}, \alpha_{ij}) \mid (i, j) \in I\}, \tag{2.2}$$

where

(1) I is a set of indices

$$I = \{(i,j),\ i = 0, 1,..., n;\ j = 0, 1,..., m(i)\} \tag{2.3}$$

with n, $m(i) \in N \cup \{0\}$, lexicographically ordered.

(2) f_i, f_{ij}, are sufficiently regular real-valued functions of R^2.

(3) u_{ij} is a point of R^2 such that

$$f_i(u_{ij}) = f_{ij}(u_{ij}) = 0.$$

(4) α_{ij} is a constant that can take the values 0 or 1. It is assumed that $\alpha_{im(i)} = 1$ $i = 0, 1,..., n$, although this restriction is not essential

(5) If $f_i(u_{hk}) = 0$ and $\alpha_{hk} = 1$ with $i \leqslant h$, then

$$\nabla f_i|_{u_{hk}} = \operatorname{grad} f_i|_{u_{hk}} \neq 0.$$

(6) If $f_{ij}(u_{ik}) = 0$ and $\alpha_{ik} = 1$ with $j \leqslant k$, then

$$\nabla f_{ij}|_{u_{ik}} \neq 0.$$

(7) If $\alpha_{ij} = 1$, the vectors $\nabla f_i|_{u_{ij}}$, $\nabla f_{ij}|_{u_{ij}}$ are linearly independent.

Remark. It is possible to have $f_{i_1} = f_{i_2}$ for $i_1 \neq i_2$, $f_{i_1 j_1} = f_{i_2 j_2}$ for $(i_1, j_1) \neq (i_2, j_2)$, etc. (See examples below.)

EXAMPLE 1. $I = \{(0,0), (0,1), (1,0), (1,1)\}$.

$$f_0 = xy^2 - x^2 \begin{cases} f_{00} = y - 2, & u_{00} = (0,2), \ \alpha_{00} = 1 \\ f_{01} = y - 1, & u_{01} = (1,1), \ \alpha_{01} = 1 \end{cases}$$

$$f_1 = x + y - 2 \begin{cases} f_{10} = x, & u_{10} = (0,2), \ \alpha_{10} = 0 \\ f_{11} = y - 2, & u_{11} = (0,2), \ \alpha_{11} = 1. \end{cases}$$

Let us denote

$$I' = \{(i,j) \in I \mid \alpha_{ij} = 1\}. \tag{2.4}$$

Both the number of interpolation data and the dimension of the interpolating space constructed below equal the cardinal of I'. We address first the construction of the space.

For every $(i,j) \in I'$ we define two integers t_{ij} and p_{ij} given by:

if $j = 0$, then $t_{i0} = p_{i0} = 0$;

if $j > 0$, then t_{ij} (resp. p_{ij}) is the number of functions f_{ih} with $h < j$ such

that $f_{ih}(u_{ij})=0$ and $\nabla f_{ih}|_{u_{ij}}$ is linearly dependent (resp. linearly independent) on the vector $\nabla f_i|_{u_{ij}}$.
Also, for every i such that $(i,j)\in I'$ we define

$$e_i = \max_{j\,|\,(i,j)\in I'} \left\{ t_{ij} + \left[\frac{p_{ij}}{2}\right] + 1 \right\}, \tag{2.5}$$

where $[z]$ denotes integer part of z. For convenience, we define $e_{-1}=0$.

DEFINITION 2. The set of functions

$$B(S)=\{\psi_{ij},\,(i,j)\in I'\}, \tag{2.6}$$

where

$$\psi_{ij}=f_0^{e_0}f_1^{e_1}\cdots f_{i-1}^{e_{i-1}}f_{i0}\cdots f_{ij-1}, \qquad (i,j)\in I', \tag{2.7}$$

is called the basis associated with S. As usual, we set $f_{-1}=f_{i,-1}=1$.

EXAMPLE 2. The basis associated with the system of the example 1 is

$$B(S)=\{1,\, y-2,\, (xy^2-x^2)x\},$$

since

$$I'=\{(0,0),\,(0,1),\,(1,1)\}$$

and $t_{00}=p_{00}=t_{01}=p_{01}=0$, $e_0=1$. We denote $\mathscr{B}(S)=\operatorname{span} B(S)$. Once more, for every $(i,j)\in I'$ we write:

if $j=0$ then $T_{i0}=P_{i0}=0$;

if $j>0$ then T_{ij} (resp. P_{ij}) is the sum of the e_h such that $h<i$, $f_h(u_{ij})=0$ and $\nabla f_h|_{u_{ij}}$ is linearly dependent (resp. linearly independent) on the vector $\nabla f_i|_{u_{ij}}$.

We set

$$\rho_{\bar{i}j}=\left(-\frac{\partial f_i}{\partial y},\,\frac{\partial f_i}{\partial x}\right)\Bigg|_{u_{ij}}, \tag{2.8}$$

$$\rho_{i\bar{j}}=\left(-\frac{\partial f_{ij}}{\partial y},\,\frac{\partial f_{ij}}{\partial x}\right)\Bigg|_{u_{ij}}. \tag{2.9}$$

DEFINITION 3. The set of linear forms

$$\mathscr{L}(S)=\{L_{ij}\}_{(i,j)\in I'}, \tag{2.10}$$

where

$$L_{ij}(f)=\left.\frac{\partial^{T_{ij}+t_{ij}+P_{ij}+p_{ij}}f}{\partial\rho_{ij}^{T_{ij}+t_{ij}}\,\partial\rho_{\bar{i}j}^{P_{ij}+p_{ij}}}\right|_{u_{ij}}, \tag{2.11}$$

is called the set of data $\mathscr{L}(S)$ associated with the system S.

EXAMPLE 3. In the example 1 we have

$$T_{00}=P_{00}=t_{00}=p_{00}=0$$
$$T_{01}=P_{01}=t_{01}=p_{01}=0$$
$$T_{11}=t_{11}=0,\ P_{11}=p_{11}=1, \qquad \rho_{\bar{1}1}=(-1,1).$$

Hence

$$\mathscr{L}(S)=\{L_{00},L_{01},L_{11}\},$$

with

$$L_{00}(f)=f(0,2),\ L_{01}(f)=f(1,1),$$

and

$$L_{11}(f)=\frac{\partial^2 f}{\partial\rho_{\bar{1}1}^2}(0,2)=\left(-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 f(0,2)$$
$$=\left.\left(\frac{\partial^2 f}{\partial x^2}-2\,\frac{\partial^2 f}{\partial x\,\partial y}+\frac{\partial^2 f}{\partial y^2}\right)\right|_{(0,2)}.$$

Later, the elements of $\mathscr{L}(S)$ are applied to the functions in $B(S)$. Therefore $L_{ij}(\psi_{hk})$ must make sense. Since ψ_{hk} is a product of functions f_i, f_{rs}, it could be required to differentiate them at some points u_{hk}, thus rendering necessary the regularity assumed above.

3. Interpolation Problem Associated with an Interpolation System in R^2

Let S be a system in R^2. We associate with S the following interpolation problem

$$p\in\mathscr{B}(S),\ L_{ij}(p)=z_{ij} \qquad \forall(i,j)\in I' \tag{3.1}$$

where $L_{ij}\in\mathscr{L}(S)$ and $z_{ij}\in R\ \forall(i,j)\in I'$.

THEOREM 1. *For any interpolation system in R^2, the determinant*

$$det(L_{ij}(\psi_{hk}))_{(i,j),(h,k)\in I'}. \tag{3.2}$$

with rows and columns lexicographically ordered, is triangular and different from zero.

Proof. The proof uses repeated, careful application of the Leibniz formula

$$\frac{\partial^\alpha}{\partial\rho^\alpha}\prod_{i=1}^{n} f_i = \sum_{\beta=|\alpha|}\frac{\alpha!}{\beta_1!\beta_2!\cdots\beta_n!}\prod_{i=1}^{n}\frac{\partial^{\beta_i} f_i}{\partial\rho^{\beta_i}} \tag{3.3}$$

and cannot be reproduced here because of its length. The interested reader is referred to [6].

COROLLARY 1. *$B(S)$ is a basis for $\mathscr{B}(S)$ and the linear functionals L_{ij} are linearly independent on $\mathscr{B}(S)$.*

Remark. Of course, the theorem remains true if $\rho_{\bar{i}j}$, ρ_{ij} are replaced by any nonnull scalar multiple. Thus, for example (4, 0) is ussually replaced by (1, 0), (2, 2) by (1, 1), etc.

EXAMPLE 4. In Example 1

$$\det L_{ij}(\psi_{hk}) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

COROLLARY 2. *For any $z=(z_{ij}\mid(i,j)\in I')\in R^{\mathrm{card}\, I'}$, the problem* (3.1) *has a unique solution. This solution can be obtained by solving a triangular linear system.*

EXAMPLE 5. In Example 1, if $z_{00}=1$, $z_{01}=0$, $z_{11}=-1$, we have $a_{00}=1$, $a_{01}=1$, $a_{11}=-1/4$.

4. A FORMULA FOR THE REMAINDER

Let us consider $z_{ij}=L_{ij}(f)$. Then the problem (3.1) consists of finding $\psi\in\mathscr{B}(S)$ such that

$$L_{ij}(\psi)=L_{ij}(f) \qquad \forall (i,j)\in I' \tag{4.1}$$

and we say that ψ is the interpolating function of f in the space $\mathscr{B}(S)$ for the problem (3.1). The difference $f-\psi$ is called the remainder of the interpolating function

$$E(S;f)(x,y)=f(x,y)-\psi(x,y). \tag{4.2}$$

The coefficient a_{ij} of ψ_{ij} in ψ can be denoted by

$$a_{ij}=\left[\begin{array}{c|c} \{\psi_{hk},(h,k)\in I'\} & f \\ \{L_{hk},(h,k)\in I'\} & (i,j) \end{array}\right] \tag{4.3}$$

and called [2, 6] divided difference (i,j) of f with respect to the problem (3.1).

We have the formula

$$a_{ij}=\frac{L_{ij}(f)-L_{ij}(h_{rs})}{L_{ij}(\psi_{ij})}, \tag{4.4}$$

where (i,j) is the element which follows (r,s) in I', and

$$h_{rs}=\sum_{\substack{(p,t)\in I' \\ (p,t)\leqslant(r,s)}} a_{pt}\psi_{pt}. \tag{4.5}$$

Thus a_{ij} does not depend on either the element of $B(S)$ or the elements of $\mathscr{L}(S)$ with index greater than (i,j). Therefore instead of (4.3), we can employ the alternative notations

$$a_{ij}=\left[\begin{array}{c|c} \{\psi_{hk},(h,k)\in I',(h,k)\leqslant(i,j)\} & f \\ \{L_{hk},(h,k)\in I',(h,k)\leqslant(i,j)\} & (i,j) \end{array}\right]. \tag{4.6}$$

Let us call

$$V\left[\begin{array}{c} \{\{\psi_{hk},(h,k)\in I',(h,k)<(i,j)\},f\} \\ \{L_{hk},(h,k)\in I',(h,k)\leqslant(i,j)\} \end{array}\right] \tag{4.7}$$

the determinant which results from the application of

$$\{L_{hk},\ (h,k)\in I',\ (h,k)\leqslant(i,j)\}$$

lexicographically ordered to the functions

$$\{\{\psi_{hk},\ (h,k)\in I',\ (h,k)<(i,j)\},f\},$$

where f is the last element.

With this notations we can write

$$\psi(x,y)=\sum_{(i,j)\in I'}\left[\begin{array}{c|c} \{\psi_{hk},(h,k)\in I',(h,k)\leqslant(i,j)\} & f \\ \{L_{hk},(h,k)\in I',(h,k)\leqslant(i,j)\} & (i,j) \end{array}\right]\psi_{ij}(x,y) \tag{4.8}$$

and also

$$\psi(x,y)=\sum_{(i,j)\in I'}\frac{1}{\prod_{(h,k)\in I',(h,k)<(i,j)}L_{hk}(\psi_{hk})} \times V\begin{bmatrix}\{\{\psi_{hk},(h,k)\in I',(h,k)<(i,j)\},f\}\\ \{L_{hk},(h,k)\in I',(h,k)\leqslant(i,j)\}\end{bmatrix}\psi_{ij}(x,y). \tag{4.9}$$

Let $(\bar{x},\bar{y})$ be the point at which we wish to compute $E(S;f)$, and let us assume that f_i, $i=0,\dots,n$ do not vanish at $(\bar{x},\bar{y})$. We construct another system $\bar{S}$ adjointing to S the element

$$(f_{n+1},f_{n+1,0},u_{n+1,0},\alpha_{n+1,0})=(x-\bar{x},y-\bar{y},(\bar{x},\bar{y}),1), \tag{4.10}$$

leading to

$$\left.\begin{aligned}\bar{\psi}_{ij}&=\psi_{ij}\leftrightarrow\forall(i,j)\in I'\\ \bar{\psi}_{n+1,0}&=f_0^{e_0}f_1^{e_1}\cdots f_n^{e_n}\\ \bar{L}_{ij}&=L_{ij}\leftrightarrow\forall(i,j)\in I'\\ \bar{L}_{n+1,0}(f)&=f(\bar{x},\bar{y})\end{aligned}\right\} \tag{4.11}$$

and hence

$$B(\bar{S})=B(S)\cup\{\bar{\psi}_{n+1,0}\} \tag{4.12}$$

$$\mathscr{L}(\bar{S})=\mathscr{L}(S)\cup\{\bar{L}_{n+1,0}\}. \tag{4.13}$$

Let $\bar{\psi}$ be the interpolating function of f associated with $\bar{S}$. We have

$$\bar{\psi}=\psi+a_{n+1,0}\psi_{n+1,0}, \tag{4.14}$$

where

$$L_{ij}(\psi)=L_{ij}(f)\qquad\forall(i,j)\in I'. \tag{4.15}$$

Since

$$\bar{L}_{n+1,0}(\bar{\psi})=\bar{\psi}(\bar{x},\bar{y})=\bar{L}_{n+1,0}(f)=f(\bar{x},\bar{y}),$$

we can write

$$\begin{aligned}E(S;f)(\bar{x},\bar{y})&=f(\bar{x},\bar{y})-\psi(\bar{x},\bar{y})=\bar{\psi}(\bar{x},\bar{y})-\psi(\bar{x},\bar{y})\\ &=a_{n+1,0}\bar{\psi}_{n+1,0}(\bar{x},\bar{y})\end{aligned}$$

and then

$$E(S;f)(\bar{x},\bar{y})=\left[\begin{array}{c|c}\{\psi_{ij},(i,j)\in I',\bar{\psi}_{n+1,0}\} & f \\ \{L_{ij},(i,j)\in I',L_{n+1,0}\} & (n+1,0)\end{array}\right]\psi_{n+1}(\bar{x},\bar{y}) \qquad (4.16)$$

or also

$$E(S;f)(\bar{x},\bar{y})=\frac{1}{\prod_{(i,j)\in I'}L_{ij}(\psi_{ij})}\,V\left[\begin{array}{c}\{\{\psi_{ij},(i,j)\in I'\},f\} \\ \{\{L_{ij},(i,j)\in I'\},L_{n+1,0}\}\end{array}\right]. \qquad (4.17)$$

Formula (4.16) is a generalization of the error formula $f[x_0,\dots,x_n,\bar{x}]\prod_{i=0}^{n}(\bar{x}-x_i)$ in one variable, and formula (4.17) can be written also in the form

$$\frac{V\left[\begin{array}{c}\{\{\psi_{ij},(i,j)\in I'\},f\} \\ \{\{L_{ij},(i,j)\in I'\},L_{n+1,0}\}\end{array}\right]}{V\left[\begin{array}{c}\{\psi_{ij},(i,j)\in I'\} \\ \{L_{ij},(i,j)\in I'\}\end{array}\right]}, \qquad (4.18)$$

which is another version of the same formula, which uses determinants, and is well known in one variable.

These expressions hold except at the points at which some of the f_i vanish. In this case, if f and ψ_{ij}'s are continuous in R^2 it is possible to obtain $E(S;f)(x,y)$ by continuity.

5. Some Particular Cases in which the Exponents e_i Can Be Decreased

There are some particular cases in which the value of e_i can be decreased. For example, let us consider the case

$$f_i=a_ix+b_iy+c_i, \qquad |a_i|+|b_i|>0, \qquad \forall i<n. \qquad (5.1)$$

The function f_n is not required to be of this form.

Let us now denote, for $i=0, 1,\dots, n$,

$$e_i=\max_{j\,|\,(i,j)\in I'}\{t_{ij}+1\}. \qquad (5.2)$$

Then we have

THEOREM 2. *Let f_i, $\forall i<n$, be as in* (5.1). *Then, theorem* 1 *remains true when e_i is given by* (5.2) *rather than by* (2.5).

The proof is quite similar to that of Theorem 1.

Let us remark that, in systems of the type (5.1), if every f_{ij}, $j = 0,\dots, m(i) - 1$ (when $m(i) > 1$), has gradient linearly independent from that of f_i at every u_{ik}, $(i, k) \in I'$, then

$$e_i = 1 \qquad \forall i.$$

The systems considered in [3] are a particular case of these systems, with f_{ij} also as in (5.1). In [3] one had $I' = I$, that is $\alpha_{ij} = 1$ $\forall (i, j)$.

Theorem 2 solves a large number of interpolation problems in R^2 satisfying the following condition:

Condition H. If the value of one derivative of order s of f at a point $u \in R^2$ is an interpolation datum, then the other s derivatives of f of the order s at u, must also be given as interpolation data.

This condition is satisfied by most of the usual interpolation problems in finite element methods.

The proof of this fact can be found in [6]. We now give an example that cannot be solved with the tools given in [3], but can be solved by using the present techniques because it satisfies condition H.

EXAMPLE 6. Let us consider three non aligned points $A_i = (x_i, y_i)$ in R^2. Any interpolation problem with a set of data

$$\left\{ \left.\frac{\partial f}{\partial x}\right|_{A_i}, \left.\frac{\partial f}{\partial y}\right|_{A_i}, i = 1, 2, 3, ; \left.\frac{\partial^2 f}{\partial x\, \partial y}\right|_{A_1}, \left.\frac{\partial^2 f}{\partial y^2}\right|_{A_1}, \left.\frac{\partial^2 f}{\partial x^2}\right|_{A_1} \right\} \tag{5.3}$$

verifies condition H.

Let r_1, r_2, r_3 be

$$r_1(x, y) = (y - y_1)(x_2 - x_1) - (x - x_1)(y_2 - y_1)$$
$$r_2(x, y) = (y - y_2)(x_3 - x_2) - (x - x_2)(y_3 - y_2)$$
$$r_3(x, y) = (y - y_3)(x_1 - x_3) - (x - x_3)(y_1 - y_3).$$

If we take

$$I = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 0), (1, 1),$$
$$(1, 2), (2, 0), (2, 1), (2, 2), (3, 0)\}$$
$$f_0 = r_1, \qquad f_1 = r_2, \qquad f_2 = r_3, \qquad f_3 = r_1$$
$$f_{00} = f_{01} = f_{02} = r_3, \qquad f_{03} = f_{04} = r_2$$
$$u_{00} = u_{01} = u_{02} = A_1, \qquad u_{02} = u_{04} = A_2$$

$$\alpha_{00}=0,\qquad \alpha_{01}=\alpha_{02}=1,\qquad \alpha_{03}=0,\qquad \alpha_{04}=1$$
$$f_{10}=r_1\qquad f_{11}=f_{12}=r_3$$
$$u_{10}=A_2\qquad u_{11}=u_{12}=A_3$$
$$\alpha_{01}=1\qquad \alpha_{11}=0,\qquad \alpha_{12}=1$$
$$f_{20}=r_2\qquad f_{21}=f_{22}=r_1$$
$$u_{20}=A_3$$
$$\alpha_{20}=1\qquad \alpha_{21}=\alpha_{22}=1$$
$$f_{30}=r_3$$
$$u_{30}=A_1$$
$$\alpha_{30}=1$$

we have a system S with $\mathscr{L}(S)$ equivalent to (5.3) as it is easily seen. The interpolating space $\mathscr{B}(S)$ is a subspace of dimension 9 of the space of polynomials of degree $\leqslant 4$.

Of course, the functions f_i, f_{ij} are usually polynomials, and one would like to get $\mathscr{B}(S)=\Pi_n$, space of polynomials of degree non greater than n. One has the following result:

THEOREM 3. *Let S be a system with f_i, f_{ij} polynomials, $\alpha_{ij}=1\ \forall (i,j)\in I$ and* card $I=(n+1)(n+2)/2$. *Then $\mathscr{B}(S)=\Pi_n$ iff f_i, f_{ij} are linear functions*

$$f_i(x,y)=a_ix+b_iy+c_i\qquad \forall i<n$$
$$f_{ij}(x,y)=a_{ij}x+b_{ij}y+c_{ij},\qquad i=0,1,\ldots,n,\ j<m(i),$$

$m(i)=n-i\ \forall i$, and e_i is given by (5.2) *for $i<n$.*

Remark. The theorem would not be true without the assumption that f_i, f_{ij} are polynomials. In particular it does not hold if f_i, f_{ij} are allowed to be rational functions ([6]).

6. Interpolation Systems in R^1 and R^3

The theory above can be easily adapted to cover interpolation in one variable: we define an interpolation system in R^1 as a set of triples

$$S=\{(f_i,x_i,\alpha_i)\mid i\in I\},\tag{6.1}$$

where

(1) $I = \{0, 1, \ldots, n\}$;

(2) f_i, $i = 0, \ldots, n$ are sufficiently regular real valued functions of R.

(3) x_i is a point of R such that $f_i(x_i) = 0$.

(4) α_i is a constant that takes the values 0 or 1.

(5) If $f_i(x_j) = 0$ and $\alpha_j = 1$ with $i \leqslant j$ then

$$\frac{df_i}{dx}(x_j) \neq 0.$$

We now denote:

$$I' = \{j \in I \mid \alpha_j = 1\}.$$

For every $i \in I'$ we define an integer number t_i:

if $i > 0$, then t_i is the number of functions f_j, with $j < i$, such that $f_j(x_i) = 0$,

if $i = 0$, then $t_0 = 0$.

We can define the associated basis $B(S)$

$$B(S) = \{\psi_i\}_{i \in I'} \tag{6.2}$$

with

$$\psi_i = f_0 \cdot f_1 \cdots f_{i-1} \qquad (f_{-1} = 1) \tag{6.3}$$

and the associated set of data

$$\mathscr{L}(S) = \{L_i\}_{i \in I'} \tag{6.4}$$

with

$$L_i(f) = \left.\frac{d^{t_i} f}{dx^{t_i}}\right|_{x_i}.$$

In this setting the result of theorem 1 holds. The proof is very simple.

Example 7. When

$$f_i(x) = x - x_i, \qquad i < n,$$

and

$$\alpha_i = 1, \qquad i = 0, \ldots, n,$$

one has

$$\mathscr{B}(S) = \Pi_n$$

and the method provides the usual Newton formula, even for coincident points x_i.

EXAMPLE 8. We now present an example, concerning trigonometrical interpolation. Let $x_0, \ldots, x_{2m} \in [a, a + 2\pi)$ be $2m + 1$ points. Let us assume that

$$x_i - x_j \neq \pi \qquad \forall i, j \in I = \{0, 1, \ldots, 2m\}. \tag{6.5}$$

Take

$$f_i(x) = \sin(x - x_i), \qquad i = 0, 2, 4, \ldots, 2m,$$

$$f_i(x) = \frac{\sin((x - x_i)/2)}{\cos((x - x_{i-1})/2)}, \qquad i = 1, 3, \ldots, 2m - 1,$$

and $\alpha_i = 1\ \forall i$. If all the points are different, the resulting problem is that of Lagrange. If a point appears s times, we have s derivatives (till$(s - 1)$th order) as data at the point (Hermite interpolation). $\mathscr{B}(s)$ is the space of trigonometrical polynomials of order m, $P_m(\sin x, \cos x)$.

We have the following Newton-like formula for the solution of the trigonometrical Hermite (or Lagrange) interpolation problem:

$$\begin{aligned} p(x) = a_0 + a_1 \sin(x - x_0) + a_2 \sin \frac{x - x_0}{2} \cdot \sin \frac{x - x_1}{2} \\ + a_3 \sin \frac{x - x_0}{2} \sin \frac{x - x_1}{2} \sin(x - x_2) + \cdots \\ + a_{2m-1} \left(\prod_{i=0}^{2m-3} \sin \frac{x - x_i}{2} \right) \sin(x - x_{2m-2}) + a_{2m} \prod_{i=0}^{2m-1} \sin \frac{x - x_i}{2}, \end{aligned} \tag{6.6}$$

where the coefficients a_i are easily computable by recurrence. By taking an even number $(2m)$ of data it is possible to get a space $\mathscr{B}(S)$ such that

$$P_{m-1}(\sin x, \cos x) \subset \mathscr{B}(S) \subset P_m(\sin x, \cos x).$$

The method can also be applied in R^3. We would consider sets

$$S = \{(f_i, f_{ij}, f_{ijk}, u_{ijk}, \alpha_{ijk}), (i, j, k) \in I\}. \tag{6.7}$$

The two-dimensional results are extended without any further theoretical difficulty [6], but the notations become more cumbersome.

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